Stirling numbers

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- **1st kind:** $\left[{n \atop k}\right]$:= # of n–permutations with k cycles. (n cycle k)
  (beads) (necklaces)
- **2nd kind:** $\left\{{n \atop k}\right\}$:= # of partitions of an n–element set into k subsets. (n subset k)

- $\left\{{4 \atop 2}\right\} = 7$, $\{1,2,3,4\} = \{1,2,3\}\cup\{4\}$, $\{1,2,4\}\cup\{3\}$, $\{1,3,4\}\cup\{2\}$, $\{2,3,4\}\cup\{1\}$,
  $\{1,2\}\cup\{3,4\}$, $\{1,3\}\cup\{2,4\}$, $\{1,4\}\cup\{2,3\}$;

  $\left[{4 \atop 2}\right] = 11$, [1,2,3] [4], [1,2,4] [3], [1,3,4] [2], [2,3,4] [1],
  [1,3,2] [4], [1,4,2] [3], [1,4,3] [2], [2,4,3] [1],
  [1,2] [3,4], [1,3] [2,4], [1,4] [2,3];

- Recurrences

$$\left\{{n \atop k}\right\} = k\left\{{n-1 \atop k}\right\} + \left\{{n-1 \atop k-1}\right\}.$$

$$\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right].$$

Permutation (bijective functions)

$$\begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8\,9\,10 \\ 3\,8\,4\,7\,2\,9\,1\,5\,6\,10 \end{pmatrix} \equiv [1,3,4,7]\,[2,8,5]\,[6,9]\,[10]$$

↑ 11

- Special values $(n>0)$

$$\left\{{n \atop 0}\right\} = \left[{n \atop 0}\right] = 0.$$

$$\left\{{n \atop 1}\right\} = 1 \qquad \left[{n \atop 1}\right] = (n-1)!$$

$$\left\{{n \atop 2}\right\} = (2^{n-1}-1) \qquad \left[{n \atop 2}\right] = (n-1)!\,H_{n-1}$$

$$\left\{{n \atop n-1}\right\} = \left[{n \atop n-1}\right] = \binom{n}{2}.$$

$$\left\{{n \atop n}\right\} = \left[{n \atop n}\right] = \binom{n}{n} = 1.$$

$$\left\{{n \atop k}\right\} = \left[{n \atop k}\right] = \binom{n}{k} = 0, \quad \text{if } k > n.$$

n	$\left\{{n \atop 0}\right\}$	$\left\{{n \atop 1}\right\}$	$\left\{{n \atop 2}\right\}$	$\left\{{n \atop 3}\right\}$	$\left\{{n \atop 4}\right\}$	$\left\{{n \atop 5}\right\}$	$\left\{{n \atop 6}\right\}$
0	1						
1	0	1					
2	0	1	1				
3	0	1	3	1			
4	0	1	7	6	1		
5	0	1	15	25	10	1	
6	0	1	31	90	65	15	1

n	$\left[{n \atop 0}\right]$	$\left[{n \atop 1}\right]$	$\left[{n \atop 2}\right]$	$\left[{n \atop 3}\right]$	$\left[{n \atop 4}\right]$	$\left[{n \atop 5}\right]$	$\left[{n \atop 6}\right]$
0	1						
1	0	1					
2	0	1	1				
3	0	2	3	1			
4	0	6	11	6	1		
5	0	24	50	35	10	1	
6	0	120	274	225	85	15	1

- $\left\{{n \atop 2}\right\} = \frac{1}{2}\left[\binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1}\right] = \frac{1}{2}\left[2^n - 2\right]$ #

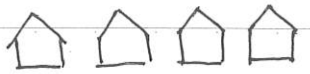
$[n, a_{n-1}, \ldots, a_{k+1}]\,[a_k, a_{k-1}, \ldots, a_1]$

- $\left[{n \atop 2}\right] = (n-1)\left[{n-1 \atop 2}\right] + (n-2)!$

$$\frac{\left[{n \atop 2}\right]}{(n-1)!} = \frac{\left[{n-1 \atop 2}\right]}{(n-2)!} + \frac{1}{n-1} = \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{\left[{2 \atop 2}\right]}{1!}$$
$$= H_{n-1}$$

或 $\left[{n \atop 2}\right] = \sum_{1\le k\le n-1} (n-1)\cdots(k+1)\,\frac{k(k-1)\cdots 1}{k}$

$$= (n-1)!\sum_{1\le k\le n-1}\frac{1}{k} = (n-1)!\,H_{n-1}$$



- Converting between powers:

$$x^{n} = \sum_k \left\{ n \atop k \right\} x^{\underline{k}} = \sum_k \left\{ n \atop k \right\} (-1)^{n-k} x^{\overline{k}}.$$

$$x^{\underline{n}} = \sum_k \left[n \atop k \right] (-1)^{n-k} x^{k};$$

$$x^{\overline{n}} = \sum_k \left[n \atop k \right] x^{k}.$$

$$x^4 = x^{\underline{4}} + 6x^{\underline{3}} + 7x^{\underline{2}} + x^{\underline{1}} = \left\{ 4 \atop 4 \right\} x^{\underline{4}} + \left\{ 4 \atop 3 \right\} x^{\underline{3}} + \left\{ 4 \atop 2 \right\} x^{\underline{2}} + \left\{ 4 \atop 1 \right\} x^{\underline{1}}$$

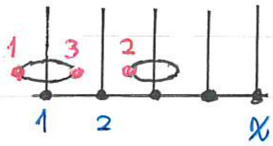
$$x^{\underline{4}} = x(x-1)(x-2)(x-3) = x^4 - 6x^3 + 11x^2 - 6x = \left[4 \atop 4 \right] x^4 - \left[4 \atop 3 \right] x^3 + \left[4 \atop 2 \right] x^2 - \left[4 \atop 1 \right] x$$

$$x^{\overline{4}} = x(x+1)(x+2)(x+3) = x^4 + 6x^3 + 11x^2 + 6x = \left[4 \atop 4 \right] x^4 + \left[4 \atop 3 \right] x^3 + \left[4 \atop 2 \right] x^2 + \left[4 \atop 1 \right] x$$

- Inversion formulas:

$$\sum_k \left[n \atop k \right] \left\{ k \atop m \right\} (-1)^{n-k} = [m=n];$$

$$\sum_k \left\{ n \atop k \right\} \left[k \atop m \right] (-1)^{n-k} = [m=n].$$



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$$\begin{bmatrix} x^4 \\ x^3 \\ x^2 \\ x^1 \\ x^0 \end{bmatrix} = \begin{bmatrix} \left\{ 4 \atop 4 \right\} & \left\{ 4 \atop 3 \right\} & \left\{ 4 \atop 2 \right\} & \left\{ 4 \atop 1 \right\} & \left\{ 4 \atop 0 \right\} \\ 0 & \left\{ 3 \atop 3 \right\} & \left\{ 3 \atop 2 \right\} & \left\{ 3 \atop 1 \right\} & \left\{ 3 \atop 0 \right\} \\ 0 & 0 & \left\{ 2 \atop 2 \right\} & \left\{ 2 \atop 1 \right\} & \left\{ 2 \atop 0 \right\} \\ 0 & 0 & 0 & \left\{ 1 \atop 1 \right\} & \left\{ 1 \atop 0 \right\} \\ 0 & 0 & 0 & 0 & \left\{ 0 \atop 0 \right\} \end{bmatrix} \begin{bmatrix} x^{\underline{4}} \\ x^{\underline{3}} \\ x^{\underline{2}} \\ x^{\underline{1}} \\ x^{\underline{0}} \end{bmatrix}, \quad \begin{bmatrix} x^{\overline{4}} \\ x^{\overline{3}} \\ x^{\overline{2}} \\ x^{\overline{1}} \\ x^{\overline{0}} \end{bmatrix} = \begin{bmatrix} \left[4 \atop 4 \right] & \left[4 \atop 3 \right] & \left[4 \atop 2 \right] & \left[4 \atop 1 \right] & \left[4 \atop 0 \right] \\ 0 & \left[3 \atop 3 \right] & \left[3 \atop 2 \right] & \left[3 \atop 1 \right] & \left[3 \atop 0 \right] \\ 0 & 0 & \left[2 \atop 2 \right] & \left[2 \atop 1 \right] & \left[2 \atop 0 \right] \\ 0 & 0 & 0 & \left[1 \atop 1 \right] & \left[1 \atop 0 \right] \\ 0 & 0 & 0 & 0 & \left[0 \atop 0 \right] \end{bmatrix} \begin{bmatrix} x^4 \\ x^3 \\ x^2 \\ x^1 \\ x^0 \end{bmatrix}$$

$$\begin{bmatrix} x^{\underline{4}} \\ x^{\underline{3}} \\ x^{\underline{2}} \\ x^{\underline{1}} \\ x^{\underline{0}} \end{bmatrix} = \begin{bmatrix} \left[4 \atop 4 \right] & -\left[4 \atop 3 \right] & \left[4 \atop 2 \right] & -\left[4 \atop 1 \right] & \left[4 \atop 0 \right] \\ 0 & \left[3 \atop 3 \right] & -\left[3 \atop 2 \right] & \left[3 \atop 1 \right] & -\left[3 \atop 0 \right] \\ 0 & 0 & \left[2 \atop 2 \right] & -\left[2 \atop 1 \right] & \left[2 \atop 0 \right] \\ 0 & 0 & 0 & \left[1 \atop 1 \right] & -\left[1 \atop 0 \right] \\ 0 & 0 & 0 & 0 & \left[0 \atop 0 \right] \end{bmatrix} \begin{bmatrix} x^4 \\ x^3 \\ x^2 \\ x^1 \\ x^0 \end{bmatrix}$$

$$\begin{bmatrix} \left\{ 4 \atop 4 \right\} & \left\{ 4 \atop 3 \right\} & \left\{ 4 \atop 2 \right\} & \left\{ 4 \atop 1 \right\} & \left\{ 4 \atop 0 \right\} \\ 0 & \left\{ 3 \atop 3 \right\} & \left\{ 3 \atop 2 \right\} & \left\{ 3 \atop 1 \right\} & \left\{ 3 \atop 0 \right\} \\ 0 & 0 & \left\{ 2 \atop 2 \right\} & \left\{ 2 \atop 1 \right\} & \left\{ 2 \atop 0 \right\} \\ 0 & 0 & 0 & \left\{ 1 \atop 1 \right\} & \left\{ 1 \atop 0 \right\} \\ 0 & 0 & 0 & 0 & \left\{ 0 \atop 0 \right\} \end{bmatrix} \begin{bmatrix} \left[4 \atop 4 \right] & -\left[4 \atop 3 \right] & \left[4 \atop 2 \right] & -\left[4 \atop 1 \right] & \left[4 \atop 0 \right] \\ 0 & \left[3 \atop 3 \right] & -\left[3 \atop 2 \right] & \left[3 \atop 1 \right] & -\left[3 \atop 0 \right] \\ 0 & 0 & \left[2 \atop 2 \right] & -\left[2 \atop 1 \right] & \left[2 \atop 0 \right] \\ 0 & 0 & 0 & \left[1 \atop 1 \right] & -\left[1 \atop 0 \right] \\ 0 & 0 & 0 & 0 & \left[0 \atop 0 \right] \end{bmatrix} = I_5$$

$$\begin{bmatrix} \left[4 \atop 4 \right] & -\left[4 \atop 3 \right] & \left[4 \atop 2 \right] & -\left[4 \atop 1 \right] & \left[4 \atop 0 \right] \\ 0 & \left[3 \atop 3 \right] & -\left[3 \atop 2 \right] & \left[3 \atop 1 \right] & -\left[3 \atop 0 \right] \\ 0 & 0 & \left[2 \atop 2 \right] & -\left[2 \atop 1 \right] & \left[2 \atop 0 \right] \\ 0 & 0 & 0 & \left[1 \atop 1 \right] & -\left[1 \atop 0 \right] \\ 0 & 0 & 0 & 0 & \left[0 \atop 0 \right] \end{bmatrix} \begin{bmatrix} \left\{ 4 \atop 4 \right\} & \left\{ 4 \atop 3 \right\} & \left\{ 4 \atop 2 \right\} & \left\{ 4 \atop 1 \right\} & \left\{ 4 \atop 0 \right\} \\ 0 & \left\{ 3 \atop 3 \right\} & \left\{ 3 \atop 2 \right\} & \left\{ 3 \atop 1 \right\} & \left\{ 3 \atop 0 \right\} \\ 0 & 0 & \left\{ 2 \atop 2 \right\} & \left\{ 2 \atop 1 \right\} & \left\{ 2 \atop 0 \right\} \\ 0 & 0 & 0 & \left\{ 1 \atop 1 \right\} & \left\{ 1 \atop 0 \right\} \\ 0 & 0 & 0 & 0 & \left\{ 0 \atop 0 \right\} \end{bmatrix} = I_5$$

(6.15) $\underbrace{\{\cdots\}_1\{\cdots\}_2\cdots\{\cdots\}_m}_{k}\underbrace{\{\cdots,0\}_{m+1}}_{n-k}$ $\quad\begin{cases} n=12 \\ m=2 \end{cases}$ $k=5$

(6.16) $[0\ 9\ 12\ 7\ 10\ 1\ 2\ 11][3\ 5][4\ 8\ 6] \Longleftrightarrow [1\ 2\ 11][3\ 5][4\ 8\ 6][7\ 10][9\ 12]$

Table 265 Additional Stirling number identities, for integers $l, m, n \geqslant 0$.

$$\left\{{n+1 \atop m+1}\right\} = \sum_k \binom{n}{k}\left\{{k \atop m}\right\}. \qquad (6.15)$$

$$\left[{n+1 \atop m+1}\right] = \sum_k \left[{n \atop k}\right]\binom{k}{m}. \qquad (6.16)$$

$$\left\{{n \atop m}\right\} = \sum_k \binom{n}{k}\left\{{k+1 \atop m+1}\right\}(-1)^{n-k}. \qquad (6.17)$$

$$\left[{n \atop m}\right] = \sum_k \left[{n+1 \atop k+1}\right]\binom{k}{m}(-1)^{m-k}. \qquad (6.18)$$

$$m!\left\{{n \atop m}\right\} = \sum_k \binom{m}{k}k^n(-1)^{m-k}. \qquad (6.19)$$

$$\left\{{n+1 \atop m+1}\right\} = \sum_{k=0}^{n} \left\{{k \atop m}\right\}(m+1)^{n-k}. \qquad (6.20)$$

$$\left[{n+1 \atop m+1}\right] = \sum_{k=0}^{n} \left[{k \atop m}\right]n^{\underline{n-k}} = n!\sum_{k=0}^{n}\left[{k \atop m}\right]/k!. \qquad (6.21)$$

$$\left\{{m+n+1 \atop m}\right\} = \sum_{k=0}^{m} k\left\{{n+k \atop k}\right\}. \qquad (6.22)$$

$$\left[{m+n+1 \atop m}\right] = \sum_{k=0}^{m} (n+k)\left[{n+k \atop k}\right]. \qquad (6.23)$$

$$\binom{n}{m} = \sum_k \left\{{n+1 \atop k+1}\right\}\left[{k \atop m}\right](-1)^{m-k}. \qquad (6.24)$$

$$n^{\underline{n-m}}[n \geqslant m] = \sum_k \left[{n+1 \atop k+1}\right]\left\{{k \atop m}\right\}(-1)^{m-k}. \qquad (6.25)$$

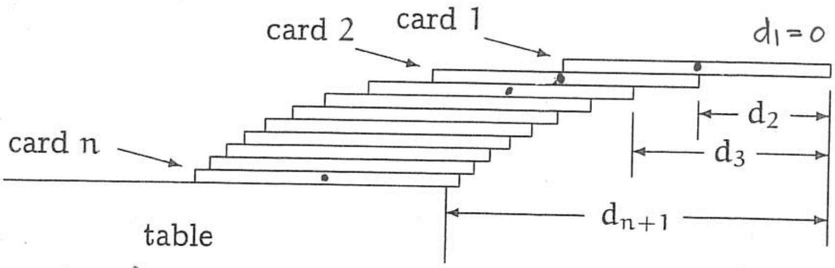
$$\left\{{n \atop n-m}\right\} = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left[{m+k \atop k}\right]. \qquad (6.26)$$

$$\left[{n \atop n-m}\right] = \sum_k \binom{m-n}{m+k}\binom{m+n}{n+k}\left\{{m+k \atop k}\right\}. \qquad (6.27)$$

$$\left\{{n \atop l+m}\right\}\binom{l+m}{l} = \sum_k \left\{{k \atop l}\right\}\left\{{n-k \atop m}\right\}\binom{n}{k}. \qquad (6.28)$$

$$\left[{n \atop l+m}\right]\binom{l+m}{l} = \sum_k \left[{k \atop l}\right]\left[{n-k \atop m}\right]\binom{n}{k}. \qquad (6.29)$$

Harmonic numbers



- $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$

- **Brick stacking:**

$$\left.\begin{aligned} 1 &\geq d_2 \\ \frac{1+(1+d_2)}{2} &\geq d_3 \\ \frac{1+(1+d_2)+(1+d_3)}{3} &\geq d_4 \\ &\vdots \\ \frac{(1+d_1)+(1+d_2)+\cdots+(1+d_k)}{k} &\geq d_{k+1} \end{aligned}\right\} \quad (1\leq k\leq n)$$

$$\begin{aligned} k\,d_{k+1} &= d_1+\cdots+d_k+k \\ (k-1)\,d_k &= d_1+\cdots+d_{k-1}+k-1 \\ \hline k\,d_{k+1} &= k\,d_k+1 \end{aligned}$$

$$\begin{cases} d_{k+1} = d_k+\frac{1}{k} \\ d_1 = 0 \end{cases} \Rightarrow d_{k+1} = H_k$$

- **Warm on the rubber band:** reach the other end? (100 cm)

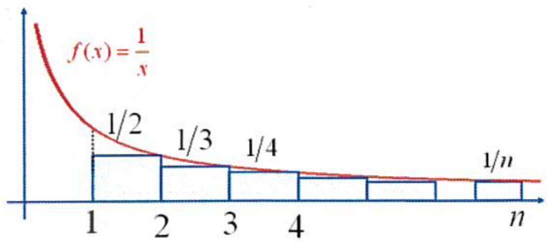
(1) 爬 1 cm, stretch 1 cm: $\frac{1}{100}+\frac{1}{101}+\cdots+\frac{1}{100+n-1} \geq 1$?

(2) 〃 , 100 cm: $\frac{1}{100}+\frac{1}{200}+\cdots+\frac{1}{100n} = \frac{1}{100}H_n \geq 1$?

- $H_n = \underbrace{\frac{1}{1}}+\underbrace{\frac{1}{2}+\frac{1}{3}}+\underbrace{\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}}+\underbrace{\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}}+\underbrace{\frac{1}{16}+\cdots+\frac{1}{n}}$ $(16\leq n<32)$

(1) $\frac{\lfloor \lg n\rfloor}{2}+1 = 1+\frac{1}{2}+2\cdot\frac{1}{4}+4\cdot\frac{1}{8}+8\cdot\frac{1}{16} < H_n < 1+2\frac{1}{2}+4\frac{1}{4}+8\frac{1}{8}+16\frac{1}{16} = \lfloor \lg n\rfloor+1$

(2) $\ln(n+1) < H_n < \ln n+1$



$$\int_1^{n+1}\frac{1}{x}dx < 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n} < 1+\int_1^n\frac{1}{x}dx = 1+\ln n$$

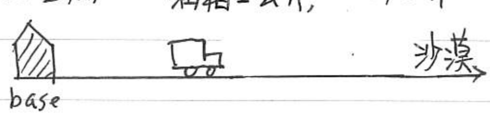
(3) $H_n = \ln n+\gamma+\frac{1}{2n}-\frac{1}{12n^2}+O\left(\frac{1}{n^4}\right)$

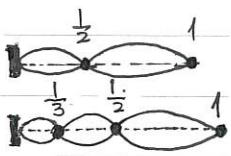
(Euler const) $\gamma = 0.57721\ldots$

(4) $H_{10^6} = 14.392$ (百萬張卡)

(5) $H_n \geq 100,\ n \geq e^{100-\gamma}$

- **Jeep problem:** 走多遠?





1 trip : 1

2 trips : $1+\frac{1}{2}$

3 〃 : $1+\frac{1}{2}+\frac{1}{3}$ $\quad 2(1-d) \geq 4d \Rightarrow d\leq\frac{1}{3}$

n 〃 : $1+\frac{1}{2}+\cdots+\frac{1}{n}$, $\quad 2(1-d_n) \geq 2(n-1)d_n \Rightarrow d_n\leq\frac{1}{n}$, can travel $H_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$

- $\sum_{0\leq k<n}\binom{k}{m}H_k = \sum_0^n\binom{k}{m}H_k\,\Delta k$ $\qquad \begin{cases} u_k = H_k & \Delta u_k = \frac{1}{k+1} \\ \Delta v_k = \binom{k}{m}, & v_k = \binom{k}{m+1} \end{cases}$

$= H_k\binom{k}{m+1}\Big|_0^n - \sum_0^n\binom{k+1}{m+1}\frac{1}{k+1}\Delta k$

$= \binom{n}{m+1}H_n - \sum_0^n\frac{1}{m+1}\binom{k}{m}\Delta k$

$= \binom{n}{m+1}H_n - \frac{1}{m+1}\binom{k}{m+1}\Big|_0^n = \binom{n}{m+1}\left(H_n-\frac{1}{m+1}\right)$

$(m=0)\ \sum_{0\leq k<n}H_k = n(H_n-1)$

$(m=1)\ \sum_{0\leq k<n}kH_k = \frac{n(n-1)}{2}\left(H_n-\frac{1}{2}\right)$

$\vdots$

Fibonacci numbers

$$\begin{cases} F_0 = 0; \\ F_1 = 1; \\ F_n = F_{n-1} + F_{n-2}, \quad \text{for } n > 1. \end{cases}$$

n	0	1	2	3	4	5	6	7	8	9	10	11
F_n	0	1	1	2	3	5	8	13	21	34	55	89

↑ 50 (under 55)

$$F(z) = F_0 + F_1 z + F_2 z^2 + \cdots = \sum_{n \geq 0} F_n z^n.$$

$$\begin{aligned} F(z) &= F_0 + F_1 z + F_2 z^2 + F_3 z^3 + F_4 z^4 + F_5 z^5 + \cdots, \\ zF(z) &= \quad F_0 z + F_1 z^2 + F_2 z^3 + F_3 z^4 + F_4 z^5 + \cdots, \\ -)\ z^2 F(z) &= \quad F_0 z^2 + F_1 z^3 + F_2 z^4 + F_3 z^5 + \cdots. \end{aligned}$$

$$F(z) - zF(z) - z^2 F(z) = z$$

$$\begin{cases} \alpha + \beta = 1 \\ \alpha \beta = -1 \end{cases} \quad x^2 - x - 1 = 0$$

$$F(z) = \frac{z}{1 - z - z^2} = \frac{A}{1 - \alpha z} + \frac{B}{1 - \beta z} = A \sum_{n \geq 0} (\alpha z)^n + B \sum_{n \geq 0} (\beta z)^n$$

$$= \sum_{n \geq 0} (A\alpha^n + B\beta^n) z^n.$$

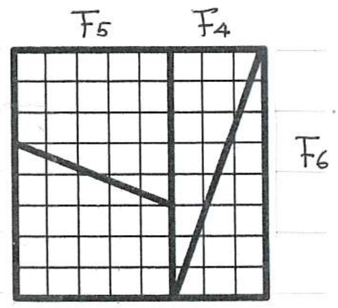
$$\begin{cases} \alpha \\ \beta \end{cases} = \frac{1 \pm \sqrt{5}}{2} \qquad \begin{cases} A \\ B \end{cases} = \pm \frac{1}{\sqrt{5}}$$

- $F_n = \frac{1}{\sqrt{5}} (\alpha^n - \beta^n) \approx \frac{1}{\sqrt{5}} \alpha^n \quad (\because |\beta| < 1)$ $\quad \left(\alpha = \frac{1+\sqrt{5}}{2} \approx 1.618\right)$ golden ratio

- 定理 (Cassini): $F_{n+1} \cdot F_{n-1} - F_n^2 = (-1)^n$, $\quad \because \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$

- 定理 $\quad F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$ $\qquad F_5 \cdot F_7 - F_6^2 = (-1)^6$

  $\Rightarrow F_n \mid F_{2n}, \; F_n \mid F_{3n}, \; \ldots, \; F_n \mid F_{kn}$

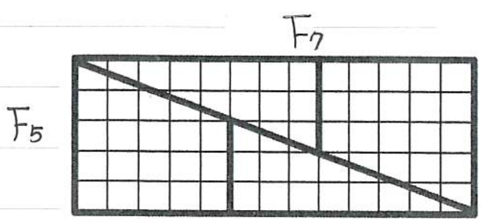


- 定理 $\quad F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$

- 定理 (Zeckendorf) $\qquad (a \gg b \Leftrightarrow a \geq b+2)$

  $n = F_{k_1} + F_{k_2} + \cdots + F_{k_r} \quad (k_1 \gg k_2 \gg \cdots \gg k_r \gg 0)$

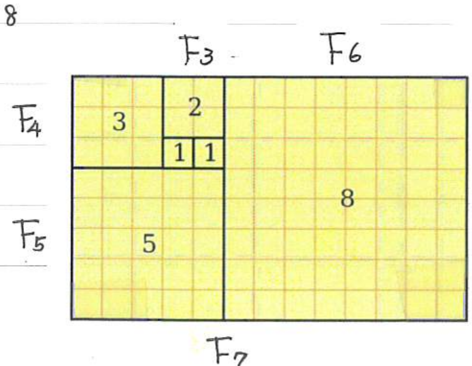
  $= (b_m b_{m-1} \cdots b_2)_F \quad$ (無連續 1)



証 $\quad F_{k_1} \leq n < F_{k_1+1} \Rightarrow n = F_{k_1} + m, \; m < F_{k_1-1}$

$F_9 \leq 50 < F_{10} \quad \Rightarrow 50 = F_9 + m, \quad m < F_{10} - F_9 = F_8$

$(\because F_8 + F_6 + F_4 + F_2 = F_9 - 1, \quad F_9 + F_7 + F_5 + F_3 = F_{10} - 1)$



$50 = F_9 + F_7 + F_4 = 34 + 13 + 3, \quad 51 = (10100101)_F$

$= (10100100)_F \qquad 52 = (10101000)_F$

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n	0	1	2	3	4
T_n	1	1	2	3	5

• **Problem** 求 T_n = # ways to pave $2 \times n$ □ with 2×1 ▯

(甲) (Recurrence) $T_n = T_{n-1} + T_{n-2}$ $(n \ge 2)$, $T(z) = \sum_{n\ge 0} T_n z^n$ (Lebesgue sum) $\quad T_n = F_{n+1}$

$$T(z) = 1 + zT(z) + z^2 T(z) \Rightarrow T(z) = \frac{1}{1-z-z^2} = \frac{1}{z}F(z)$$

(乙) Symbolic G.F. (Rieman sum)

T = 1 + ▯ + ⊞ + ⊟ + ▯⊞ + ⊞⊟ + ⊟⊞ + ⋯ $\xrightarrow{▯ \to z}$ $T(z) = 1 + z + z^2 + z^2 + z^3 + z^3 + z^3 + \cdots$

= 1 + ▯(1 + ▯ + ⊞ + ⊟ + ⋯) + ⊟(1 + ▯ + ⊞ + ⊟ + ⋯)

= 1 + ▯T + ⊟T. $\qquad = 1 + zT(z) + z^2 T(z)$

• **Remarks** (1) T_n = # ways to pave $1 \times n$ □ with 1×1 □ or 1×2 □.

(爬 n 階樓梯，每次 1步/2步)

(2) T_n = # Morse codes in n seconds $\quad \begin{cases} \cdot & 1秒 \\ — & 2秒 \end{cases}$

• **Continuants (連分多項式)**

$K_n(x_1,\ldots,x_n) = K_{n-1}(x_1,\ldots,x_{n-1})x_n + K_{n-2}(x_1,\ldots,x_{n-2})$ $(n\ge 2)$

$K_0() = 1;$ $\qquad$ ε

$K_1(x_1) = x_1;$ $\qquad$ ·

$K_2(x_1,x_2) = x_1x_2 + 1;$ $\qquad$ ·· $\quad$ —

$K_3(x_1,x_2,x_3) = x_1x_2x_3 + x_1 + x_3;$ $\qquad$ ··· $\quad$ ·— $\quad$ —·

$K_4(x_1,x_2,x_3,x_4) = x_1x_2x_3x_4 + x_1x_2 + x_1x_4 + x_3x_4 + 1$ $\qquad$ ···· $\quad$ ··— $\quad$ ·—· $\quad$ —·· $\quad$ ——

• **定理** (1) $K_n(x_1,\ldots,x_n)$ 含 $K_n(1,1,\ldots,1) = F_{n+1}$ 項

(2) $K_n(x_1,\ldots,x_n) \xrightarrow[\text{缺 } x_i x_{i+1} \to —]{x_j \to \bullet} C_n = \{n 秒 \text{ Morse Codes}\}$, $\quad C_n = C_{n-1}\bullet + C_{n-2}—$

(3) $F_{n+1} = \sum_{k=0}^{n} \binom{n-k}{k}$ $\quad \begin{cases} k 個 & — \\ n-2k 個 & \bullet \end{cases}$

(4) $K_n(x_1,\ldots,x_n) = x_1 K_{n-1}(x_2,\ldots,x_n) + K_{n-2}(x_3,\ldots,x_n)$

(5) $\frac{64}{11} = [5,1,4,2] = \frac{K_4(5,1,4,2)}{K_3(1,4,2)} = [5,1,4,1,\underbrace{1}_{1+\frac{1}{1}}] = R^5 L^1 R^4 L^1$

5	64	11	1
	55	9	
4	9	2	2
	8	2	
	1	0	

$\frac{64}{11} = 5 + \frac{9}{11} = 5 + \frac{1}{\frac{11}{9}}$, $\quad 64 = 5\cdot 11 + 9$, $\quad K_4(5,1,4,2) = 5\cdot K_3(1,4,2) + K_2(4,2)$

$= 5 + \frac{1}{1+\frac{2}{9}}$ $\quad 11 = 1\cdot 9 + 2$ $\quad K_3(1,4,2) = 1\cdot K_2(4,2) + K_1(2)$

$9 = 4\cdot 2 + 1$ $\quad K_2(4,2) = 4\cdot K_1(2) + K_0()$

$= 5 + \frac{1}{1+\frac{1}{4+\frac{1}{2}}}$ $\quad 2 = 2\cdot 1$ $\quad K_1(2) = 2$

Continued Fractions (連分數)

$$[a_0, a_1, \dots, a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}}, \quad \frac{64}{11} = 5 + \cfrac{1}{1 + \cfrac{1}{4 + \frac{1}{2}}} = [5, 1, 4, 2]$$

• **定理** $[a_0, a_1, \dots, a_n]$ 令 $\begin{cases} [a_0, \dots, a_k] = \dfrac{p_k}{q_k} \\ [a_k, \dots, a_n] = \dfrac{P_k}{Q_k} \end{cases}$

則 (1) $\begin{cases} p_{-1} = 1, & p_0 = a_0, & p_k = a_k p_{k-1} + p_{k-2} \\ q_{-1} = 0, & q_0 = 1, & q_k = a_k q_{k-1} + q_{k-2} \end{cases}$ $(1 \le k \le n)$, 且 $\begin{vmatrix} p_{k-1} & p_k \\ q_{k-1} & q_k \end{vmatrix} = (-1)^k$ $(p_k \perp q_k)$

k	-1	0	1	2	3
a_k		5	1	4	2
p_k	1	5	6	29	64
q_k	0	1	1	5	11

(2) $\begin{cases} P_{n+1} = 1, & P_n = a_n, & P_k = a_k P_{k+1} + P_{k+2} \\ & Q_n = 1, & Q_k = P_{k+1} \end{cases}$ $(0 \le k \le n-1)$

k	0	1	2	3	4
a_k	5	1	4	2	
P_k	64	11	9	2	1
Q_k	11	9	2	1	

(3) $[a_0, a_1, \dots, a_n] = \dfrac{p_n}{q_n} = \dfrac{P_0}{Q_0} = \dfrac{P_0}{P_1} = \dfrac{K_{n+1}(a_0, \dots, a_n)}{K_n(a_1, \dots, a_n)}$

證明 (Induction)

(1) $[a_0] = \dfrac{a_0}{1}$, $[a_0, a_1] = a_0 + \dfrac{1}{a_1} = \dfrac{a_1 a_0 + 1}{a_1}$, $\dfrac{p_{k+1}}{q_{k+1}} = [a_0, \dots, a_k, a_{k+1}] = [a_0, \dots, a_k + \dfrac{1}{a_{k+1}}]$

$$= \frac{\left(a_k + \frac{1}{a_{k+1}}\right) p_{k-1} + p_{k-2}}{\left(a_k + \frac{1}{a_{k+1}}\right) q_{k-1} + q_{k-2}} = \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) + p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) + q_{k-1}} = \frac{a_{k+1} p_k + p_{k-1}}{a_{k+1} q_k + q_{k-1}}$$

$$\begin{cases} \begin{vmatrix} p_{-1} & p_0 \\ q_{-1} & q_0 \end{vmatrix} = \begin{vmatrix} 1 & a_0 \\ 0 & 1 \end{vmatrix} = (-1)^0 \\ \begin{vmatrix} p_k & p_{k+1} \\ q_k & q_{k+1} \end{vmatrix} = \begin{vmatrix} p_k & a_{k+1} p_k + p_{k-1} \\ q_k & a_{k+1} q_k + q_{k-1} \end{vmatrix} = (-1)^{k+1} \end{cases}$$

(2) $[a_n] = \dfrac{a_n}{1}$, $[a_{n-1}, a_n] = \dfrac{a_{n-1} a_n + 1}{a_n}$, $\dfrac{P_{k-1}}{Q_{k-1}} = [a_{k-1}, a_k, \dots, a_n] = a_{k-1} + \dfrac{1}{\frac{P_k}{Q_k}} = \dfrac{a_{k-1} P_k + P_{k+1}}{P_k}$

• **Remark**

解整數方程式: $192x + 33y = 15$

$(192, 33) = 3 \mid 15$, $64x + 11y = 5$, $64 \cdot 5 - 11 \cdot 29 = 1$

$64 \cdot 25 - 11 \cdot 145 = 5$